—Chapter 1—

Vector Calculus

1-1 Vector Fields

OS:

David Guichard, Vector Calculus
https://www.whitman.edu/mathematics/calculus_online/chapter16.html

A. VECTOR FUNCTIONS

(1) Parametric equations and curves

Consider the equation of a circle:

$$x^2 + y^2 = r^2$$

We will never be able to write the equation above down as a single equation of the form $y = f(x)$.

$$y = \sqrt{r^2 - x^2} \qquad \text{(top)}$$

$$y = -\sqrt{r^2 - x^2} \qquad \text{(bottom)}$$

We, thus, introduce parametric equations, defining both x and y in terms of a third variable called a parameter as follows:

$$x = f(t), \qquad y = g(t)$$

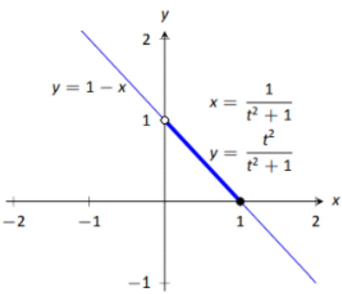
Each value of t defines a point $(x, y) = (f(t), g(t))$.

EXAMPLES:

1. Sketch the curve for the following set of parametric equations.

$$x = \frac{1}{t^2+1}, \qquad y = \frac{t^2}{t^2+1}$$

ANSWER:



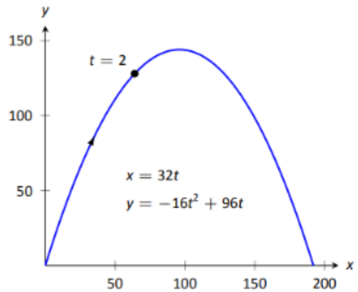
Alternative,

$$y = \frac{t^2}{t^2+1} = 1 - \frac{1}{t^2+1} = 1 - x$$

(2) A vector expression of the form $(x, y) = (f(t), g(t))$ is called a vector function.

EXAMPLES:

1. Assuming ideal projectile motion $g = 16/3$, the height of the object can be described by $y = -x^2/64 + 3x$. Describe the trajectory.



Parametric variable t

$$y = -\frac{1}{2}gt^2 = 96 \Rightarrow t = 6$$

$$x = v_x t = 192 \Rightarrow v_x = 32$$

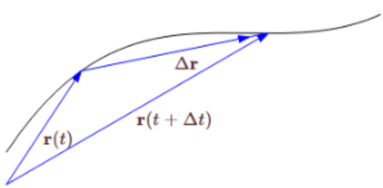
Thus, we obtain

$$x = 32t$$

$$y = -\frac{(32t)^2}{64} + 3(32t) = -16t^2 + 96t$$

The trajectory is $\vec{r} = (32t, -16t^2 + 96t)$.

(3) Calculus with vector functions



$$
\begin{aligned}
\vec{r}' &= \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left(\frac{f(t+\Delta t) - \vec{r}(t)}{\Delta t}, \frac{g(t+\Delta t) - \vec{r}(t)}{\Delta t} \right) \\
&= \left(f'(t), g'(t) \right)
\end{aligned}
$$

B. VECTOR FIELDS

(1) Each point (x, y, z) in a space indicates a vector $\vec{F}(x, y, z)$.

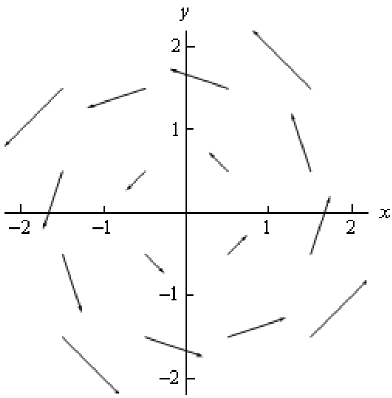
$$\vec{F}(x, y, z) = P(x, y, z)\hat{x} + Q(x, y, z)\hat{y} + R(x, y, z)\hat{z}$$

where $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ are called scalar functions.

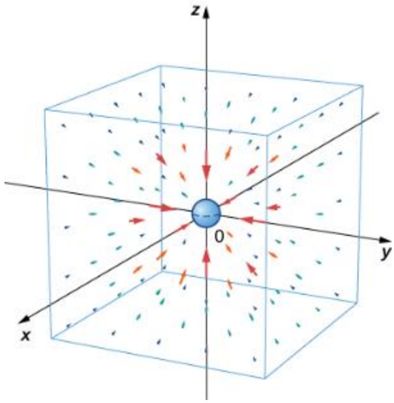
EXAMPLES:

1. Sketch the following vector fields:

$$\vec{F}(x, y) = -y\hat{x} + x\hat{y}$$



$$\vec{F} = -k \frac{e^2}{r^2} \hat{r}$$



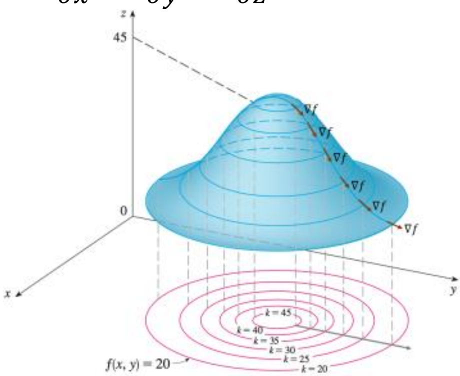
(2) Gradient field

If f is a scalar function of x, y and z, then the gradient of f is

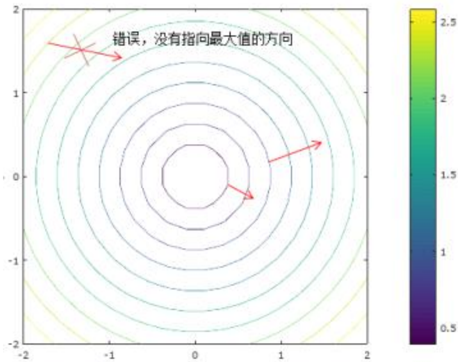
$$\nabla f(x,y,z)=\frac{\partial f(x,y,z)}{\partial x}\hat{x}+\frac{\partial f(x,y,z)}{\partial y}\hat{y}+\frac{\partial f(x,y,z)}{\partial z}\hat{z}$$

where

$$\nabla=\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}$$



The gradient vector field $\nabla f(x,y,z)$ is perpendicular to the level curves (contour) $f(x,y,z)=c$, i.e., ∇f points to the maximum rate of change at a point on a scalar function.

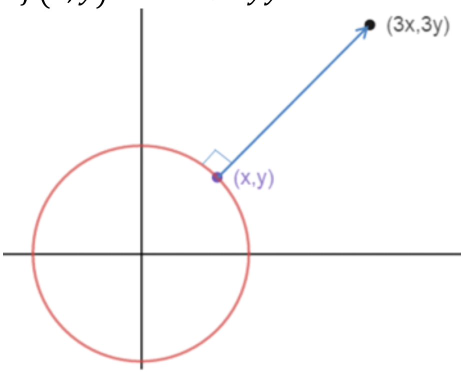


EXAMPLES:

1. Sketch the level curve of $f(x,y) = x^2 + y^2 = r$ and gradient vector field.

   ANSWER:

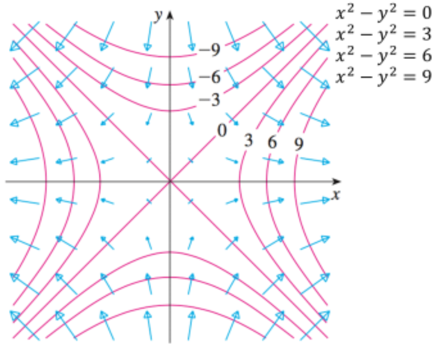
   $\nabla f(x,y) = 2x\hat{x} + 2y\hat{y}$



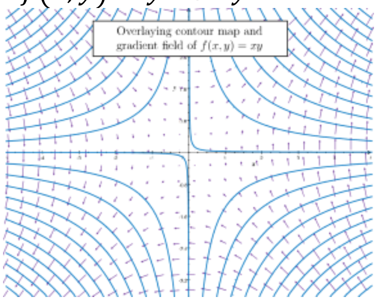
2. Sketch the level curves of $f(x,y) = x^2 - y^2$ and gradient vector fields.

   ANSWER:

   $\nabla f(x,y) = 2x\hat{x} - 2y\hat{y}$



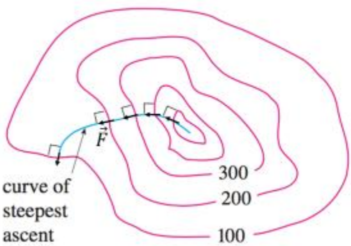
3. Sketch the level curves of $f(x,y) = xy$ and gradient vector fields.
ANSWER:
$\nabla f(x,y) = y\hat{x} + x\hat{y}$



(3) A vector field $\vec{F}(x,y,z) = P(x,y,z)\hat{x} + Q(x,y,z)\hat{y} + R(x,y,z)\hat{z}$ is called a conservative vector field if it is the gradient of some scalar function, i.e.,

$$\vec{F}(x,y,z) = \nabla f(x,y,z)$$

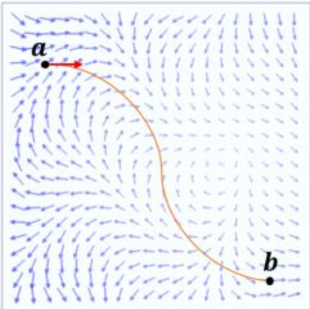
Since $\nabla f(x,y,z)$ is perpendicular to the level curves (contour) $f(x,y,z) = c$, thus, the conservative vector field $\vec{F}$ is perpendicular to the level sets of its scalar function f.



1-2 Calculus With Vector Fields

A. LINE INTEGRAL

(1) The line integral of $\vec{F}$ along the path $\mathcal{C}$



$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{s}$$

(2) Given a parameterization $\vec{s}(t)$ of the path $\mathcal{C}$

$$\vec{s}(t) = x(t)\hat{x} + y(t)\hat{y}$$

The line integral becomes

$$\int_a^b \vec{F}(\vec{s}(t)) \cdot d\vec{s} = \int_a^b \vec{F}(\vec{s}(t)) \cdot \frac{d\vec{s}(t)}{dt} dt = \int_a^b \vec{F}(\vec{s}(t)) \cdot \dot{\vec{s}}(t)\, dt$$

EXAMPLES:

1. Evaluate

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{s}$$

where $\vec{F}(x, y, z) = xz\hat{x} - yz\hat{z}$ and $\mathcal{C}$ is the line segment from $(-1,2,0)$ to $(3,0,1)$.

ANSWER:

The parameterization for the line:

$$\begin{aligned} \vec{s}(t) &= (1-\mathrm{t})(-1,2,0) + t(3,0,1) \\ &= (4t-1, 2-2t, t) \qquad , \qquad 0 \le t \le 1 \\ &= (4t-1)\hat{x} + (2-2t)\hat{y} + t\hat{z} \end{aligned}$$

$$\begin{aligned} \vec{F}(\vec{s}(t)) &= (4t-1)t\hat{x} - (2-2t)t\hat{z} \\ &= (4t^2 - t)\hat{x} - (2t - 2t^2)\hat{z} \end{aligned}$$

$$\vec{F}(\vec{s}(t)) \cdot \dot{\vec{s}}(t) = \left((4t^2 - t)\hat{x} - (2t - 2t^2)\hat{z}\right) \cdot (4\hat{x} - 2\hat{y} + \hat{z})$$
$$= 4(4t^2 - t) - (2t - 2t^2)$$
$$= 18t^2 - 6t$$
$$\int_C \vec{F} \cdot d\vec{s} = \int_0^1 (18t^2 - 6t)\, dt = 3$$

(3) If $\vec{F} = \nabla f(x, y)$, then

$$\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{s}(t)) \cdot d\vec{s}$$
$$= \int_a^b \nabla f(x, y) \cdot \dot{\vec{s}}(t)\, dt$$
$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt$$
$$= \int_a^b \frac{d}{dt} f(\vec{s}(t))\, dt$$
$$= f(\vec{s}(b)) - f(\vec{s}(a))$$

The result only depends on the initial point and final point, and is independent of path.

EXAMPLES:

1. Evaluate

$$\int_C \nabla f \cdot d\vec{s}$$

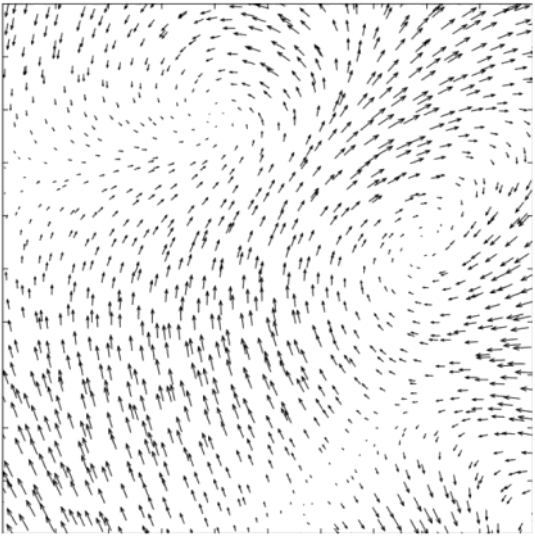
where $f(x, y, z) = \cos(\pi x) + \sin(\pi x) - xyz$ and C is any path that starts at $\left(1, \frac{1}{2}, 2\right)$ and ends at $(2,1,-1)$.

ANSWER:

$$\int_C \nabla f \cdot d\vec{s} = f(\vec{s}(b)) - f(\vec{s}(a))$$
$$= f(2,1,-1) - f\left(1, \frac{1}{2}, 2\right)$$
$$= 4$$

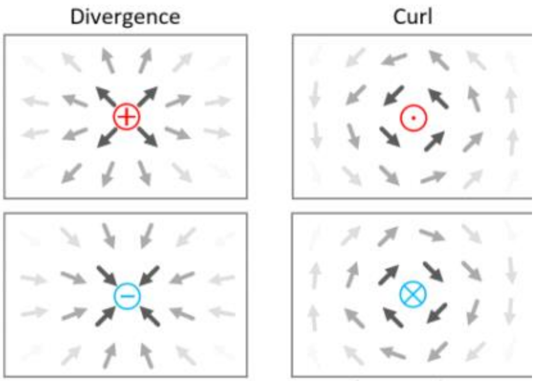
B. DIVERGENCE AND CURL

(1) A vector field visualized



(2) Vector fields characterized

Divergence and curl are two measurements of vector fields.

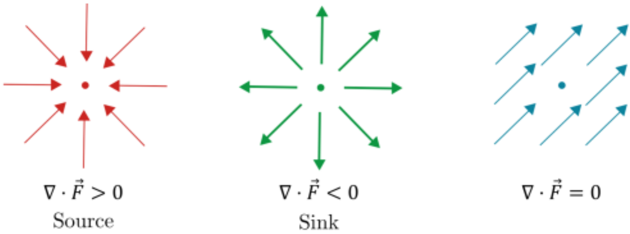


Divergence measures the tendency of the fluid to collect or disperse at a point, that is, divergence is a scalar or a single number.

Curl measures the tendency of the fluid to swirl around the point, that is, curl is itself a vector. The magnitude of the curl measures how much the fluid is swirling and the direction indicates the axis around which it tends to swirl.

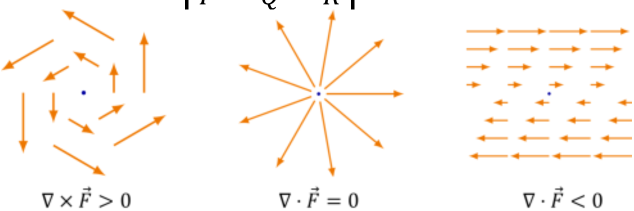
(3) The divergence of a vector field $\vec{F} = P(x,y)\hat{x} + Q(x,y)\hat{y} + R(x,y)\hat{z}$ is

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \cdot \left(P\hat{x} + Q\hat{y} + R\hat{z}\right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$



(4) The curl of a vector field $\vec{F} = P(x,y)\hat{x} + Q(x,y)\hat{y} + R(x,y)\hat{z}$ is

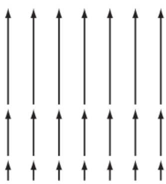
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$



EXAMPLES:

1. Sketch the vector field $\vec{F} = z\hat{z}$ and find its divergence

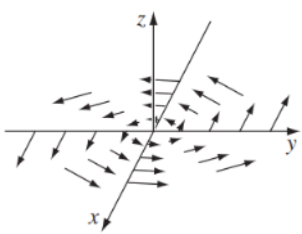
   ANSWER:

   $$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}0 + \frac{\partial}{\partial y}0 + \frac{\partial}{\partial z}z = 1$$

2. Sketch the vector fields $\vec{F}(x,y,z) = -y\hat{x} + x\hat{y}$ and find its curl

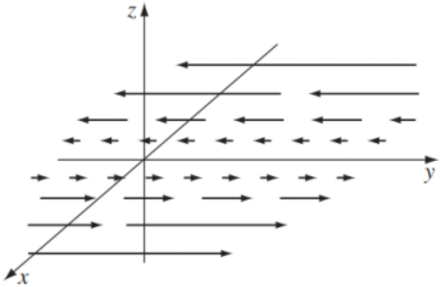
   ANSWER:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\hat{z}$$

3. Sketch the vector fields $\vec{F}(x, y, z) = x\vec{y}$ and find its curl

ANSWER:

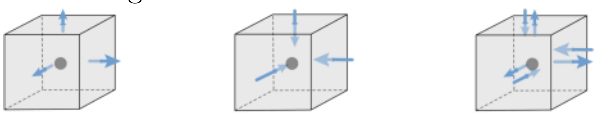


$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} = \hat{z}$$

C. CAUSS'S DIVERGENCE THEOREM AND STOKES' THEOREM

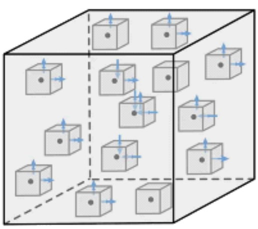
(1) Gauss's divergence theorem

The divergence in an infinitesimal volume



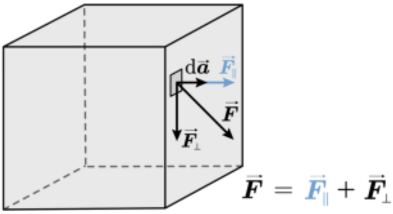
$\nabla \cdot \vec{F} > 0$ Source $\nabla \cdot \vec{F} < 0$ Sink $\nabla \cdot \vec{F} = 0$

The sum of the sources and sinks of vector fields within a volume is



$$\iiint_{\mathcal{V}} \nabla \cdot \vec{F}\, d\tau$$

Define flux of a vector field passing through an infinitesimal area:

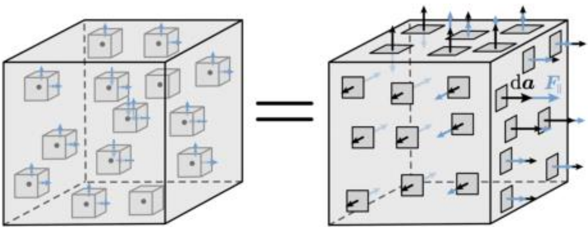


$$\vec{F} \cdot d\vec{a} = \left(\vec{F}_{\parallel} + \vec{F}_{\perp}\right) \cdot d\vec{a} = \vec{F}_{\parallel} \cdot d\vec{a} + \underbrace{\vec{F}_{\perp} \cdot d\vec{a}}_{=0} = \vec{F}_{\parallel} \cdot d\vec{a}$$

The total flux of vector fields passing through the closed surface is

$$\oiint_{\mathcal{S}} \vec{F} \cdot d\vec{a}$$

The sum of the sources and sinks of vector fields within a volume is the same as the total flux of vector fields passing through the closed surface.



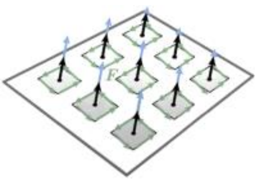
$$\iiint_{\mathcal{V}} \nabla \cdot \vec{F}\, d\tau = \oiint_{\mathcal{S}} \vec{F} \cdot d\vec{a}$$

(2) Stokes' theorem

The curl in an infinitesimal area

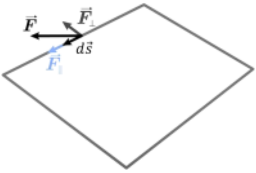
$\nabla \times \vec{F} > 0$

The sum of the curls of vector fields within an area is



$$\iint_{\mathcal{S}} \left(\nabla \times \vec{F}\right) \cdot d\vec{a}$$

Define circulation (or the amount of swirl) of a vector field along an infinitesimal loop:



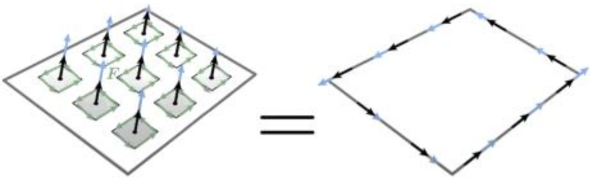
The dot-product of $\vec{F}$ along the path $d\vec{s}$

$$\vec{F} \cdot d\vec{s} = \left(\vec{F}_\parallel + \vec{F}_\perp\right) \cdot d\vec{s} = \vec{F}_\parallel \cdot d\vec{s} + \underbrace{\vec{F}_\perp \cdot d\vec{s}}_{=0} = \vec{F}_\parallel \cdot d\vec{s}$$

the total circulation of vector fields of along the closed path spanning the surface is

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{s}$$

The sum of the curls of vector fields within an area is the same as the total circulation of vector fields of along the closed path spanning the surface.



$$\iint_{\mathcal{S}} \left(\nabla \times \vec{F}\right) \cdot d\vec{a} = \oint_{\mathcal{C}} \vec{F} \cdot d\vec{s}$$

EXAMPLES:

1. Evaluate the total flux of a vector field $\vec{F} = y^2\hat{x} + \left(2xy + z^2\right)\hat{y} + 2yz\hat{z}$ go through a unit cube at the origin.

   ANSWER:

   According to Gauss's divergence theorem,

   $$\oiint_{\mathcal{S}} \vec{F} \cdot d\vec{a} = \iiint_{\mathcal{V}} \nabla \cdot \vec{F}\, d\tau$$

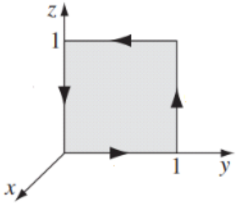
   Hence, we need to calculate

   $$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}y^2 + \frac{\partial}{\partial y}\left(2xy + z^2\right) + \frac{\partial}{\partial z}2yz = 2\left(x + y\right)$$

   Thus, we obtain

$$\iiint_{\mathcal{V}} \nabla \cdot \vec{F}\, d\tau = 2\int_0^1 \int_0^1 \int_0^1 (x+y)\, dxdydz$$
$$= 2\int_0^1 \int_0^1 \left(\frac{1}{2} + y\right)\, dydz$$
$$= 2$$

2. Evaluate the circulation of a vector field $\vec{F} = (2xz + 3y^2)\hat{y} + 4yz^2\hat{z}$ for a unit square surface at $x = 0$.



ANSWER:

According to Stokes' theorem,

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{s} = \iint_{\mathcal{S}} \left(\nabla \times \vec{F}\right) \cdot d\vec{a}$$

Hence, we need to calculate

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 2xz + 3y^2 & 4yz^2 \end{vmatrix} = (4z^2 - 2x)\hat{x} + 2z\hat{z}$$

$$d\vec{a} = dydz\hat{x}$$

Since $x = 0$ for this surface, we obtain

$$\iint_{\mathcal{S}} \left(\nabla \times \vec{F}\right) \cdot d\vec{a} = \int_0^1 \int_0^1 4z^2\, dydz = \frac{4}{3}$$

D. SECOND DERIVATIVES FOR A VECTOR FIELD

(1) The gradient of a scalar function $f(x, y, z)$ is a vector field

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

- Divergence of gradient:

$$\nabla \cdot (\nabla f) = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \cdot \left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right)$$
$$= \frac{\partial^2 f}{\partial x^2}\hat{x} + \frac{\partial^2 f}{\partial y^2}\hat{y} + \frac{\partial^2 f}{\partial z^2}\hat{z}$$
$$= \nabla^2 f$$

- Curl of gradient:

$$\nabla \times (\nabla f) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = 0$$

(2) Second derivatives for a vector field

$$\vec{F}(x,y,z) = P(x,y,z)\hat{x} + Q(x,y,z)\hat{y} + R(x,y,z)\hat{z}$$

- Divergence of curl:

$$\nabla \cdot \left(\nabla \times \vec{F}\right) = \nabla \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0$$

- Curl of curl:

$$\nabla \times \left(\nabla \times \vec{F}\right) = \nabla\left(\nabla \cdot \vec{F}\right) - \nabla^2 \vec{F}$$

1-3 Helmholtz Theorem

A. HELMHOLTZ THEOREM

(1) Let $\vec{F}(\vec{r})$ be a vector field such that

$$\nabla \cdot \vec{F} = D(r)$$
$$\nabla \times \vec{F} = \vec{C}(\vec{r})$$

If $D(r)$ and $\vec{C}(\vec{r})$ go to zero sufficiently rapidly at infinity, then, $\vec{F}(\vec{r})$ has a **unique** decomposition:

$$\vec{F} = -\nabla\varphi + \nabla \times \vec{A}$$

where

$$\varphi = \frac{1}{4\pi}\int \frac{\nabla \cdot \vec{F}}{|\vec{r} - \vec{r}'|} d^3r' = \frac{1}{4\pi}\int \frac{D(r')}{|\vec{r} - \vec{r}'|} d^3r'$$

$$\vec{A} = \frac{1}{4\pi}\int \frac{\nabla \times \vec{F}}{|\vec{r} - \vec{r}'|} d^3r' = \frac{1}{4\pi}\int \frac{\vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r'$$

So if we have theory for, or measurements of, the divergence and curl of our field of interest, $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ respectively, we can calculate $\vec{F}$.

EXAMPLES:

1. Suppose that

$$\nabla \cdot \vec{E} = \frac{\rho(r)}{\epsilon_0}$$
$$\nabla \times \vec{E} = 0$$

and $\vec{E}$ goes to zero at the infinity. Verify that $\vec{E} = -\nabla\varphi$ is uniquely determined.

ANSWER:

According to Helmholtz theorem, if $\rho(r)$ is given, the vector field $\vec{E}$ is

$$\vec{E} = -\nabla\varphi + \nabla \times \vec{A}$$

where

$$\varphi = \frac{1}{4\pi}\int \frac{\nabla \cdot \vec{E}}{|\vec{r} - \vec{r}'|} d^3r' = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{|\vec{r} - \vec{r}'|} d^3r'$$

$$\vec{A} = \frac{1}{4\pi}\int \frac{\nabla \times \vec{E}}{|\vec{r} - \vec{r}'|} d^3r' = 0$$

Thus, $\vec{E}$ is uniquely determined by

$\vec{E} = -\nabla\varphi$

2. Suppose that

$\nabla \cdot \vec{B} = 0$

$\nabla \times \vec{B} = \mu_0 \vec{J}$

and $\vec{B}$ goes to zero at the infinity. Verify that $\vec{B} = \nabla \times \vec{A}$ is uniquely determined.

ANSWER:

According to Helmholtz theorem, if $\vec{J}(\vec{r})$ is given, the vector field $\vec{B}$ is

$\vec{B} = -\nabla\varphi + \nabla \times \vec{A}$

where

$$\varphi = \frac{1}{4\pi}\int \frac{\nabla \cdot \vec{B}}{|\vec{r} - \vec{r}'|} d^3r' = 0$$

$$\vec{A} = \frac{1}{4\pi}\int \frac{\nabla \times \vec{B}}{|\vec{r} - \vec{r}'|} d^3r' = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r'$$

Thus, $\vec{B}$ is uniquely determined by

$\vec{B} = \nabla \times \vec{A}$

(2) Physical interpretation of φ

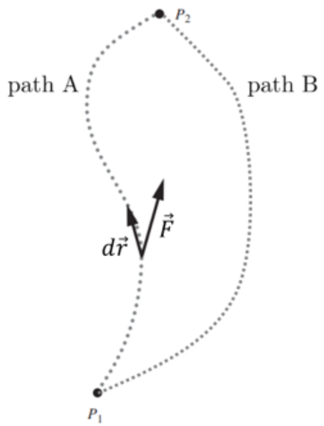
Using Stokes' theorem, we obtain

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{s} = \iint_{\mathcal{S}} \left(\nabla \times \vec{F}\right) \cdot d\vec{a}$$

For an irrotational field, i.e., $\nabla \times \vec{F} = \mathbf{0}$, thus, we have

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{s} = 0$$

Consider two paths A and B:



$$\underbrace{\int_{P_1}^{P_2} \vec{F}\cdot d\vec{s}}_{\text{path A}} + \underbrace{\int_{P_2}^{P_1} \vec{F}\cdot d\vec{s}}_{\text{path B}} = 0$$

$$\underbrace{\int_{P_1}^{P_2} \vec{F}\cdot d\vec{s}}_{\text{path A}} - \underbrace{\int_{P_1}^{P_2} \vec{F}\cdot d\vec{s}}_{\text{path B}} = 0$$

Thus, the line integral is independent of path,

$$\int_{P_1}^{P_2} \vec{F}\cdot d\vec{s} = \varphi(P_2) - \varphi(P_1)$$

Since

$$\int_{P_1}^{P_2} \nabla\varphi\cdot d\vec{s} = \varphi(P_2) - \varphi(P_1)$$

we obtain $\vec{F}$ is conservative, i.e.,

$$\vec{F} = \nabla\varphi$$

(3) Physical interpretation of $\vec{A}$

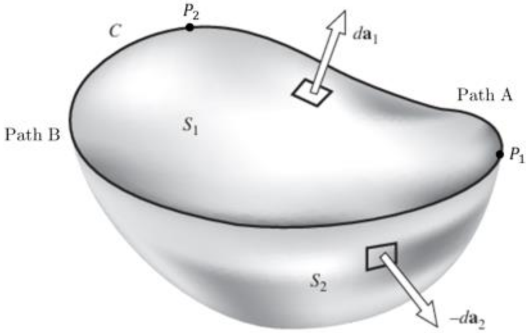
Using Gauss's divergence theorem, we obtain

$$\oiint_{\mathcal{S}} \vec{F}\cdot d\vec{a} = \iiint_{\mathcal{V}} \left(\nabla\cdot\vec{F}\right) d\tau$$

For a solenoidal field, i.e., $\nabla\cdot\vec{F} = 0$, the total flux through the closed surface is zero. Thus, we have

$$\oiint_{\mathcal{S}} \vec{F}\cdot d\vec{a} = 0$$

Consider two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$:



$$\iint_{S_1} \vec{F} \cdot d\vec{a} + \iint_{S_2} \vec{F} \cdot d\vec{a} = 0$$
$$\iint_{S_1} \vec{F} \cdot d\vec{a}_1 - \iint_{S_2} \vec{F} \cdot d\vec{a}_2 = 0$$

Thus, the surface integral is independent of the surface spanned by a closed loop $\mathcal{C}$,

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{a} = \underbrace{\int_{P_1}^{P_2} \vec{A} \cdot d\vec{s}}_{\text{path A}} - \underbrace{\int_{P_1}^{P_2} \vec{A} \cdot d\vec{s}}_{\text{path B}}$$

Since

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{s} = \underbrace{\int_{P_1}^{P_2} \vec{A} \cdot d\vec{s}}_{\text{path A}} - \underbrace{\int_{P_1}^{P_2} \vec{A} \cdot d\vec{s}}_{\text{path B}} = \iint_{\mathcal{S}} \left(\nabla \times \vec{A}\right) \cdot d\vec{a}$$

thus, we obtain

$$\vec{F} = \nabla \times \vec{A}$$

B. POISSON'S EQUATION AND LAPLACE'S EQUATION

(1) For an irrotational field, i.e., $\nabla \times \vec{F} = 0$, we have

$$\vec{F} = -\nabla\varphi$$

Thus, we can obtain

$$\nabla \cdot \vec{F} = \nabla \cdot \left(-\nabla\varphi\right) = -\nabla^2\varphi = D(r)$$

$$\begin{cases} \nabla^2\varphi = -D(r) & \cdots\cdots \text{Poisson's equation} \\ \nabla^2\varphi = 0 & \cdots\cdots \text{Laplace's equation} \end{cases}$$

where ∇^2 is called the Laplacian.

(2) For a solenoidal field, i.e., $\nabla \cdot \vec{F} = 0$, we have

$$\vec{F} = \nabla \times \vec{A}$$

Thus, we can obtain

$$\nabla \times \vec{F} = \nabla \times \nabla \times \vec{A} = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$$

Since

$$\begin{aligned}
\nabla \cdot \vec{A} &= \frac{1}{4\pi}\int\left(\nabla_r \frac{1}{\left|\vec{r}-\vec{r}'\right|}\right)\cdot \vec{C}\left(\vec{r}'\right) d^3r' \\
&= \frac{1}{4\pi}\int\left(-\nabla_{r'} \frac{1}{\left|\vec{r}-\vec{r}'\right|}\right)\cdot \vec{C}\left(\vec{r}'\right) d^3r' \\
&= -\frac{1}{4\pi}\oint_S \frac{\vec{C}\left(\vec{r}'\right)}{\left|\vec{r}-\vec{r}'\right|}\cdot d\vec{a} + \frac{1}{4\pi}\int \frac{1}{\left|\vec{r}-\vec{r}'\right|}\underbrace{\nabla_{r'}\cdot \vec{C}\left(\vec{r}'\right)}_{=0} d^3r' \\
&= -\frac{1}{4\pi}\oint_S \frac{\vec{C}\left(\vec{r}'\right)}{\left|\vec{r}-\vec{r}'\right|}\cdot d\vec{a}
\end{aligned}$$

assume that the Gaussian surface is at infinity and $\vec{C}\left(\vec{r}'\right)$ goes to zero faster than $1/r^2$ as $r \to \infty$. So the surface integral is zero. Thus, we have

$$\nabla \cdot \vec{A} = 0$$

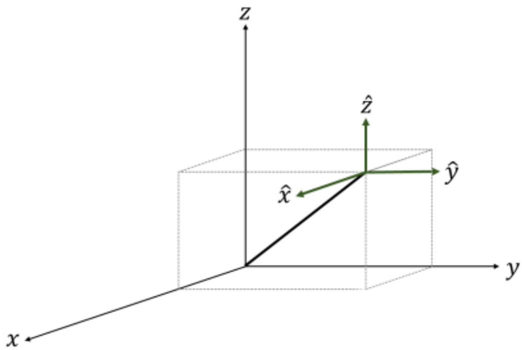
and obtain

$$\begin{cases} \nabla^2 \vec{A} = -\vec{C} \\ \nabla^2 \vec{A} = 0 \end{cases}$$

1-4 Symmetry and Curvilinear Coordinates

A. SYMMETRY OF FUNCTIONS AND VECTOR FIELDS

(1) Cartesian coordinates



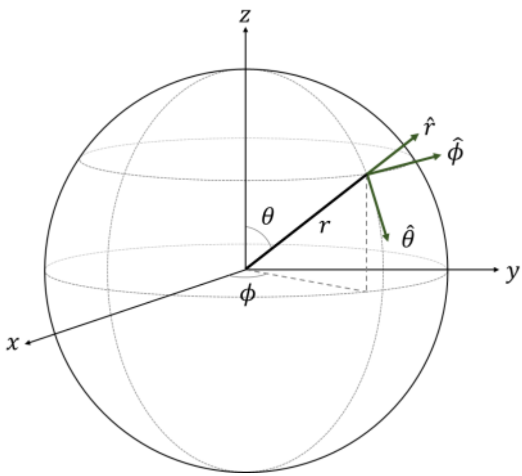
Scalar Function:

$$f(x,y,z)$$

Vector Field:

$$\vec{F}(x,y,z) = P(x,y,z)\hat{x} + Q(x,y,z)\hat{y} + R(x,y,z)\hat{z}$$

(2) Spherical symmetry and coordinates



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

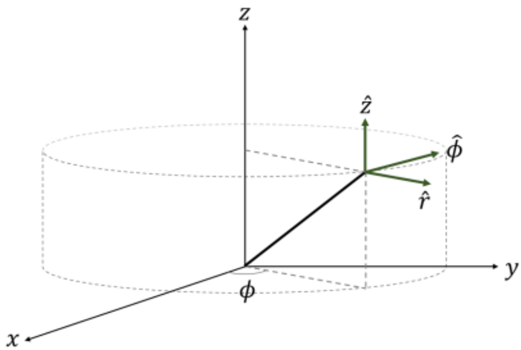
Scalar Function:

$$f(r,\theta,\phi)$$

Vector Field:

$$\vec{F}(r,\theta,\phi) = P(r,\theta,\phi)\hat{r} + Q(r,\theta,\phi)\hat{\theta} + R(r,\theta,\phi)\hat{\phi}$$

(3) Cylindrical symmetry and coordinates



$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

Scalar Function:

$$f(r,\phi,z)$$

Vector Field:

$$\vec{F}(r,\phi,z) = P(r,\phi,z)\hat{r} + Q(r,\phi,z)\hat{\phi} + R(r,\phi,z)\hat{z}$$

B. DIFFERENTIAL OPERATORS

(1) Cartesian coordinates

Gradient:

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

Divergence:

$$\nabla \cdot \vec{F}(x,y,z) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Curl:

$$\nabla \times \vec{F}(x,y,z) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Divergence of gradient = Laplacian:

$$\nabla^2 f(x,y,z) = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f$$

(2) Spherical coordinates

Gradient:

$$\nabla f(r,\theta,\phi) = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

Divergence:

$$\nabla \cdot \vec{F}(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, Q) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} R$$

Curl:

$$\nabla \times \vec{F}(r,\theta,\phi) = \begin{vmatrix} \frac{1}{r^2 \sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ P & rQ & r\sin\theta\, R \end{vmatrix}$$

Divergence of gradient = Laplacian:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) f + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) f + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} f$$

(3) Cylindrical coordinates

Gradient:

$$\nabla f(r,\phi,z) = \frac{\partial f}{\partial r}\hat{r} + \frac{\partial f}{r\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$$

Divergence:

$$\nabla \cdot \vec{F}(r,\phi,z) = \frac{1}{r}\frac{\partial}{\partial r}(rP) + \frac{1}{r}\frac{\partial}{\partial \phi} Q + \frac{\partial}{\partial z} R$$

Curl:

$$\nabla \times \vec{F}(r,\phi,z) = \begin{vmatrix} \frac{1}{r}\hat{r} & \hat{\phi} & \frac{1}{r}\hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ P & rQ & R \end{vmatrix}$$

Divergence of gradient = Laplacian:

$$\nabla^2 f(r,\phi,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) f + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} f + \frac{\partial^2}{\partial z^2} f$$

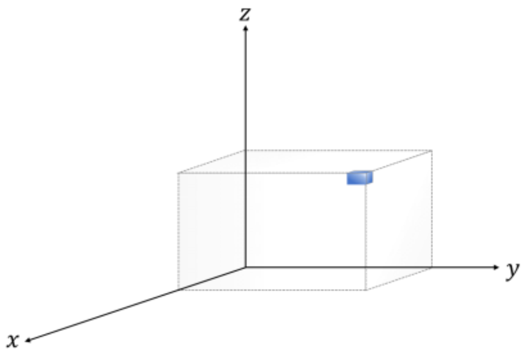
EXAMPLES:

1. Find the gradient of $1/\mathscr{r}$, where $\mathscr{r} = |\vec{r} - \vec{r}'|$.
ANSWER:

- Method I:
  In Cartesian coordinates:
  Let $\mathscr{r} = |\vec{r} - \vec{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$
  $$\nabla \frac{1}{\mathscr{r}} = -\frac{2(x-x')}{2\mathscr{r}^3}\hat{x} - \frac{2(y-y')}{2\mathscr{r}^3}\hat{y} - \frac{2(z-z')}{2\mathscr{r}^3}\hat{z} = -\frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$
- Method II:
  In spherical coordinates:
  $$\nabla \frac{1}{\mathscr{r}} = \frac{\partial}{\partial r}\left(\frac{1}{|\vec{r} - \vec{r}'|}\right)\hat{\mathscr{r}} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

C. INTEGRAL

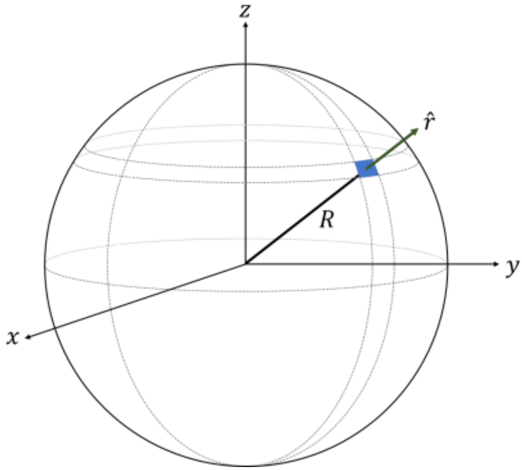
(1) Cartesian coordinates



volume elements: $d\tau = dxdydz$

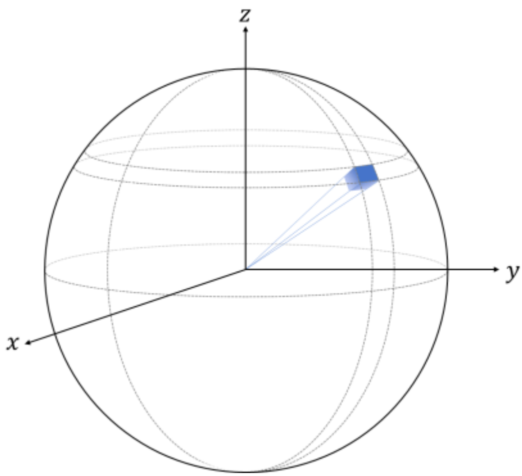
$$\iiint_{\mathcal{V}} \nabla \cdot \vec{F}\, d\tau = \iiint \nabla \cdot \vec{F}\, dxdydz$$

(2) Spherical coordinates



area elements: $da = (R\sin\theta\, d\theta)\big(Rd\phi\big) = R^2 \sin\theta\, d\theta d\phi$

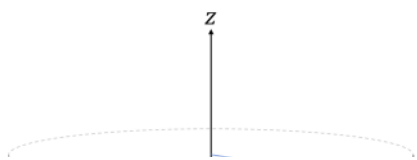
$$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{a} = \iint \vec{F}\cdot\hat{r}\, R^2 \sin\theta\, d\theta d\phi = R^2 \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \vec{F}\cdot\hat{r}\sin\theta\, d\theta d\phi$$

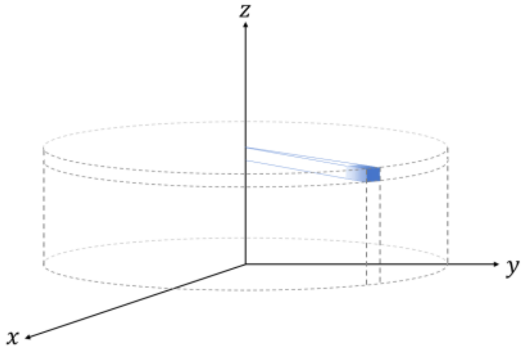


volume elements: $d\tau = (r\sin\theta\, d\theta)\big(rd\phi\big)(dr) = r^2 \sin\theta\, drd\theta d\phi$

$$\iiint_{\mathcal{V}} \nabla\cdot\vec{F}\, d\tau = \int_0^r \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \nabla\cdot\vec{F}\, r^2 \sin\theta\, drd\theta d\phi$$

(3) Cylindrical coordinates





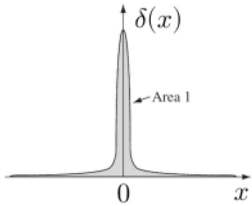
volume element: $d\tau = \left(r d\phi\right)(dz)(dr) = r dr d\phi dz$

$$\iiint_{\mathcal{V}} \nabla \cdot \vec{F}\, d\tau = \int_0^r \int_{\phi=0}^{\phi=2\pi} \int \nabla \cdot \vec{F}\, r dr d\phi dz$$

EXAMPLES:

1. Dirac delta function

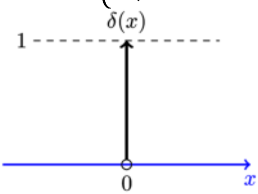
   A distribution which is well defined only when it appears under an integral sign



$$\int_a^b \delta(x)\, dx = 1, \qquad a < x < b$$

where

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$



The properties of the delta function allow us to compute

$$\int_{-\infty}^{\infty} f(x)\delta(x)\, dx = f(0)$$

Consider a vector field

$$\vec{E} = \frac{1}{r^2}\hat{r}$$

At every location, $\vec{E}$ is directed radially outward. Show that $\vec{E}$ satisfies Gauss's divergence theorem.

ANSWER:

$$\iiint_{\mathcal{V}} \nabla \cdot \vec{E}\, d\tau = \oiint_{\mathcal{S}} \vec{E} \cdot d\vec{a}$$

- R.H.S.:

  we integrate over a sphere of radius R, centered at the origin

$$\oiint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \int \frac{\hat{r}}{R^2} \cdot \hat{r}\, R^2 \sin\theta\, d\theta d\phi = \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{=2} \underbrace{\int_0^{2\pi} d\phi}_{=2\pi} = 4\pi$$

- L.H.S.:

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r} 1 = 0$$

  The divergence is zero everywhere except at the origin because as $r \to 0$, $1/r^2 \to \infty$ grows faster than $r^2 \to 0$.

  We thus define

$$\nabla \cdot \frac{1}{r^2}\hat{r} = 4\pi\delta^3(r) \text{ or equivalently } \nabla^2 \frac{1}{r} = -4\pi\delta^3(r)$$

  and obtain

$$\iiint_{\mathcal{V}} \nabla \cdot \vec{E}\, d\tau = \iiint_{\mathcal{V}} 4\pi\delta^3(r)\, d\tau = 4\pi$$